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ORIGINAL PAPER

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On the extension of the utilitarian and Suppes–Sen social welfare relations to infinite utility streams

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Abstract Extensions of a utilitarian and a Suppes–Sen grading principle defined on infinite utility streams are characterized with a stronger notion of Anonymity and without any consistency postulate. The relative merits of the Extended Utilitarian relation are discussed and its rankings are compared with those of the overtaking criterion and the Basu–Mitra Utilitarian relation.

1 Introduction

In ranking infinite utility streams, the axioms of Finite Anonymity and Strong Pareto are taken as basic guiding principles. The axiom of finite anonymity is a requirement of impartiality in the treatment of generations.[1] It insures equal treatment of generations by requiring that finite permutations of infinite utility streams do not change the social evaluation of the stream. The Strong Pareto axiom is a statement about the sensitivity of the ordering to an unidirectional change in generational utility numbers.

In a recent contribution by Mitra and Basu (2006) (henceforth, MB), the finite anonymity axiom is extended to include classes of infinite permutations such that a relation satisfying indifference to transformations of profiles in the (pre-specified) class of permutations do not come in conflict with the Strong Pareto axiom. This

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[1] See, for example, Diamond (1965) and Basu and Mitra (2003).

consistency check results in two restrictions on the extent of the allowable anonymity. If infinite permutations are allowed, then the class of permutation matrices (representing such permutations) must be (a) cyclic and (b) a group with respect to the operation of matrix multiplication.

It is shown in MB that an extended version of the Suppes–Sen grading principle satisfies the two axioms. We demonstrate that the extended Suppes–Sen grading principle is completely characterized by this extended notion of Anonymity and Strong Pareto.

Although the extended Suppes–Sen grading principle satisfies this stronger notion of anonymity, it is too conservative in making comparisons. It can only compare profiles if they are Pareto comparable or if one profile can be made to Pareto dominate the other after a suitable transformation. We propose a more complete social welfare relation (SWR)[2] preserving the Extended Anonymity axiom and allowing for a particular form of interpersonal utility comparison.[3] The allowable comparisons are restricted to comparing differences in profiles which are identical after some finite generation. The rankings of this new relation are then contrasted with two well known pre-orders, the overtaking relation [as formalized in Svensson (1980) and Asheim and Tungodden (2004)] and the Paretian type utilitarian relation (Basu and Mitra (2003)), which we call the Basu–Mitra Utilitarian relation.

The proposed relation, which we call $\mathcal{Q}$-Utilitarian, is characterized without postulating any continuity axiom[4] on the pre-order in the infinite dimensional space containing the set of utility streams. More precisely, we have characterized all pre-orders that include the $\mathcal{Q}$-Utilitarian pre-order as a subrelation. It might also be considered to be an extension of the Basu–Mitra Utilitarian relation, allowing for greater comparability between utility streams while retaining a satisfactory axiomatic basis.

We argue that the rankings by our utilitarian SWR are far more acceptable than either the overtaking relation or the Basu–Mitra Utilitarian relation. More precisely, on a class of infinite permutations (strictly larger than finite permutation) which includes the permutation in which odd periods are swapped with the adjacent even periods, the overtaking criterion ranks one utility stream (the one that starts with a higher first generation utility number) better than the permuted stream, and the Basu–Mitra utilitarian relation declares them non-comparable. On the other hand, the $\mathcal{Q}$-Utilitarian relation is indifferent when we compare the original profile with the permuted profile. The intuitive appeal of such indifference, at least with respect to this permutation, has also been argued by Van Liedekerke and Lauwers (1997). In fact more generally, on the class of profiles in which one utility stream (weakly) dominates the other after a suitable infinite permutation has been applied to it and when no finite permutation can achieve this dominance, the Basu–Mitra utilitarian relation cannot compare them, while the $\mathcal{Q}$-Utilitarian relation makes intuitive rankings.

[2] A pre-order is a transitive and reflexive binary relation. We use the term relation and pre-order interchangeably.

[3] The use of interpersonal comparability is not new to the social choice literature. With finitely many generations, possible escapes from the Arrow Impossibility theorem are obtained when interpersonal utility comparisons are allowed. See, for example, Blackorby and Bossert (2004).

[4] The continuity postulates have appeared in the form of Strong and Weak Preference Continuity. See, for example, Asheim and Tungodden (2004).

When comparing profiles in which the generational utility numbers form a summable sequence, any utilitarian rule should declare the profile which sums up to a larger number better than the other. We illustrate that even if two profiles have this property, the Basu–Mitra utilitarian relation declares them non-comparable. As a potential drawback, we highlight that the $\mathcal{Q}$-Utilitarian relation too might fail to compare between some profiles in this class.

Mathematical preliminaries and notation are defined in the next section. Section 3 summarizes the results on the extended suppes–sen grading principle and Sect. 4 is devoted to the $\mathcal{Q}$-Utilitarian relation. Section 5 considers the relative merits of the $\mathcal{Q}$ -Utilitarian relation, discusses a potential drawback and concludes the analysis. All proofs are in a separate appendix.

2 Preliminaries

2.1 Notation and definitions

Let $\mathbb{N}$ denote the set of natural numbers $\{1, 2, 3, \dots\}$ and $\mathbb{R}$ the set of real numbers. Let X denote the set $Y^{\mathbb{N}}$, where Y is the closed interval $[0, 1]$. We let X be the domain of utility sequences (also referred to as "utility streams" or "utility profiles"). Thus, we write $x \equiv (x_1, x_2, \dots) \in X$ if and only if $x_n \in [0, 1]$ for all $n \in \mathbb{N}$. Negation of a statement is indicated by the logical quantifier $\neg$.

For $y, z \in \mathbb{R}^{\mathbb{N}}$, we write $y \geq z$ if $y_i \geq z_i$ for all $i \in \mathbb{N}$ and $y > z$ if $y \geq z$ and $y \neq z$. For all $x \in X$ and all $N \in \mathbb{N}$, we denote $(x_1, \dots, x_N)$ by $x(N)$ and $(x_{N+1}, x_{N+2}, \dots)$ by $x[N]$. Thus, given any $x \in X$ and $N \in \mathbb{N}$, we can write $x = (x(N), x[N])$. For each $N \in \mathbb{N}$, the partial sum $\sum_{k=1}^{N} x_k$ is written as $I(x(N))$. When $\sum_{k=1}^{\infty} x_k < \infty$, we denote the sum by $I(x)$.

A SWR is a binary relation, $\succsim$, on X which is reflexive and transitive (a preordering). We associate with $\succsim$ its symmetric and asymmetric components in the usual way. So, we write $x \sim y$ when $x \succsim y$ and $y \succsim x$ both hold and we write $x \succ y$ when $x \succsim y$ holds, but $y \succsim x$ does not hold. A SWR $\succsim_A$ is a subrelation of SWR $\succsim_B$ if (a) $x, y \in X$, $(x \sim_A y \Rightarrow x \sim_B y)$; and (b)$x, y \in X$, $(x \succ_A y \Rightarrow x \succ_B y)$. For two SWRs $\succsim_A$ and $\succsim_B$ we write $\succsim_A \equiv \succsim_B$ iff $\succsim_A$ is a subrelation of $\succsim_B$ and $\succsim_B$ is a subrelation of $\succsim_A$.

2.2 Permutations

In this section, we briefly summarize the well-known notion of permutations. This section borrows heavily from MB. A permutation π is a one-to-one map from $\mathbb{N}$ onto $\mathbb{N}$. For any $x \in X$ and a permutation π, we write $x \circ \pi = (x_{\pi(1)}, x_{\pi(2)}, \dots) \in X$. Permutations can be represented by a permutation matrix. A permutation matrix $P = (p_{ij})_{i,j \in \mathbb{N}}$ is an infinite matrix satisfying the following properties

(a) For each $i \in \mathbb{N}$, there is some $j(i) \in \mathbb{N}$ such that $p_{ij(i)} = 1$ and $p_{ij} = 0$ for all $j \neq j(i)$.
(b) For each $j \in \mathbb{N}$, there is some $i(j) \in \mathbb{N}$ such that $p_{i(j)j} = 1$ and $p_{ij} = 0$ for all $i \neq i(j)$.

Writing permutations in terms of mappings or matrices, unsurprisingly, turns out to be equivalent. Given any permutation π, there is a permutation matrix P such that for $x \in X$, $x \circ \pi = (x_{\pi(1)}, x_{\pi(2)}, \dots)$ can also be written as Px in the usual matrix multiplication. Conversely, given any permutation matrix P, there is a permutation π defined by $\pi = Pa$, where $a = (1, 2, 3, \dots)$. The identity matrix is an infinite permutation matrix such that $p_{ii} = 1$ for all $i \in \mathbb{N}$. Given any infinite permutation matrix P, we denote by P' its unique inverse which satisfies $PP' = P'P = I$. We denote the set of all permutations (permutation matrices) by $\mathcal{P}$.

A finite permutation π is a permutation such that there is some $N \in \mathbb{N}$ with $\pi(n) = n$ for all $n > N$. Thus, a finite permutation matrix has $p_{ii} = 1$ for all $i > N$ for some $N \in \mathbb{N}$. The set of all finite permutations is denoted by $\mathcal{F}$.

We note that the entire class of permutations is a group under the usual matrix multiplication. A permutation is cyclic if for each $e_i = (0, \dots, 0, 1, 0)$ (with 1 at the ith place), there exists a $k \in \mathbb{N}$ such that $\pi^k(e_i) = e_i$. The special class of cyclic permutations is not necessarily a group. We do not discuss the mathematical structure of this class here as it is tangential to the focus of the paper. Interested readers are referred to MB for a comprehensive treatment of the class of cyclic permutation matrices. However, we present a particular class of permutations (different from $\mathcal{F}$) which is both cyclic and defines a group with respect to matrix multiplication.

Given a permutation matrix $P \in \mathcal{P}$ and $n \in \mathbb{N}$, we denote the $n \times n$ matrix $(p_{ij})_{i,j \in \{1,\dots,n\}}$ by $P(n)$. We also let P_n denote, for each $n \in \mathbb{N}$, the matrix $(p_{ij})_{i,j \in \{n+1,n+2,\dots\}}$. Let $\mathcal{S} = \{P \in \mathcal{P}$: there is some $k \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $P(nk)$ is a finite dimensional permutation matrix$\}$. This class of permutations was introduced in Lauwers (1997). It is easily checked that this class of cyclic permutations is a group (with respect to matrix multiplication).

3 Extended Suppes–Sen grading relation

In this section, we completely characterize an infinite dimensional extension of the finite Suppes–Sen grading principle. The axioms of Finite Anonymity, Strong Pareto and $\mathcal{Q}$-Anonymity are introduced first.

Finite anonymity *For all $x, y \in X$, if there is some $i, j \in \mathbb{N}$ such that $x_i = y_j$, $y_i = x_j$ and $x_t = y_t$ for all $t \neq i, j$, then $x \sim y$.*

Let $\mathcal{Q}$ be some fixed group of cyclic permutations satisfying $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{P}$. Consider the following two axioms.

$\mathcal{Q}$-**Anonymity** *For all $x \in X$, if $Q \in \mathcal{Q}$, then $Qx \sim x$.*

Strong Pareto *For all $x, y \in X$, if $x > y$, then $x \succ y$.*

Observe that $\mathcal{F}$-Anonymity is equivalent to Finite Anonymity.

Following MB, define the following extension of the Suppes–Sen grading principle[5] for all $x, y \in X$,

$$x \succsim_{QS} y \text{ if and only if there is some } P \in \mathcal{Q} \text{ such that } Px \geq y.$$

[5] The finite generation version of the grading principle is due to Suppes (1966). For a comprehensive analysis of it, see Sen (1971).

In MB it was shown that the SWR $\succsim_{QS}$ satisfies the axioms of $\mathcal{Q}$-Anonymity and Strong Pareto. We state a lemma and a proposition from MB which we use in our characterization theorems.

Lemma 1 (Mitra and Basu (2006), Lemma 1) *A permutation $Q \in \mathcal{Q}$ is cyclic iff there is no $x \in X$ satisfying $Qx > x$.*

Proposition 1 (Mitra and Basu (2006), Proposition 3) *If $\mathcal{Q}$ is a group of cyclic permutations with $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{P}$, then the SWR $\succsim_{QS}$ satisfies Strong Pareto and $\Pi(\succsim_{QS}) = \mathcal{Q}$, where*

$$\Pi(\succsim_{QS}) = \{P \in \mathcal{P} : \ Px \sim_{QS} x \quad forallx \in X\}.$$

It is proved here that any SWR satisfying $\mathcal{Q}$-Anonymity and Strong Pareto must have $\succsim_{QS}$ as its subrelation.

Proposition 2 *If $\mathcal{Q}$ is a group of cyclic permutations with $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{P}$, then $\succsim$ is a SWR on X satisfying $\mathcal{Q}$-Anonymity and Strong Pareto iff $\succsim_{QS}$ is a subrelation of $\succsim$.*

We can strengthen the conclusion of Proposition 2 further. To that effect let Ξ denote the set of all SWRs on X satisfying $\mathcal{Q}$-Anonymity and Strong Pareto. This set is non-empty as $\succsim_{QS} \in \Xi$. Consider the following binary relation: For all $x, y \in X$

$$x \succsim^* y \text{ iff } x \succsim y \quad \text{for all } \succsim \in \Xi.$$

Theorem 1 *If $\mathcal{Q}$ is a group of cyclic permutations with $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{P}$, then $\succsim^*$ is a SWR on X satisfying $\mathcal{Q}$-Anonymity and Strong Pareto. Moreover, $\succsim^* \equiv \succsim_{QS}$.*

This indicates that our analysis actually generates a result stronger than the one suggested by Proposition 2. We have established that $\mathcal{Q}$ -Anonymity and Strong Pareto generate the SWR $\succsim_{QS}$. The analysis presented here is carried out using a slightly different notation in Basu and Mitra (2003).

4 Extended utilitarian relation

The Extended Suppes–Sen relation is too restrictive. It can only compare between profiles which are themselves Pareto ordered or can be Pareto ordered after a transformation by a suitable infinite permutation. In this section, a more complete relation is proposed in the spirit of utilitarianism and its characterization is obtained. The proposed relation can be looked upon as being the least restrictive utilitarian relation that has both the Basu–Mitra utilitarian relation and extended Suppes–Sen grading principle as subrelations.

Unless otherwise mentioned, we restrict $\mathcal{Q}$ to be a group of cyclic permutations. Recall that $\mathcal{S} = \{P \in \mathcal{P}$: there is some $k \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $P(nk)$ is a finite dimensional permutation matrix$\}$. We impose a further restriction on $\mathcal{Q}$, $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{S}$. Let us first define the Basu-Mitra utilitarian relation.

We define for all $x, y \in X$,

$$\begin{gathered} x \succsim_U y \text{ if and only if} \\ \exists N \in \mathbb{N} \text{ such that } (I(x(N)), x[N]) \geq (I(y(N)), y[N]). \end{gathered} \tag{1}$$

Using the welfare relation $\succsim_U$ and our restriction on the class of permutation matrices $\mathcal{Q}$, we can define a $\mathcal{Q}$ -utilitarian relation as follows: for all $x, y \in X$,

$$x \succsim_{QU} y \text{ if and only if there is some } P \in \mathcal{Q} \text{ such that } Px \succsim_U y. \tag{2}$$

It is shown in the Appendix that $\succsim_{QU}$ is a SWR.

To obtain a complete characterization of the $\succsim_{QU}$, we need to make the idea of allowable interpersonal utility comparisons precise. This is done in the next axiom.

Partial translation scale invariance For all $x, y \in X$, $\alpha \in \mathbb{R}^{\mathbb{N}}$ and $N \in \mathbb{N}$, if $\succsim$ satisfies

$$(x(N), x[N]) \succsim (y(N), x[N]) \tag{3}$$

and

$$(x(N), x[N]) + \alpha \in X \text{ and } (y(N), x[N]) + \alpha \in X,$$

then it must also satisfy

$$(x(N), x[N]) + \alpha \succsim (y(N), x[N]) + \alpha. \tag{4}$$

We now state the Translation Scale Invariance axiom from finite population social choice theory in our notation. To that effect let us fix some $N \in \mathbb{N}$. A SWR $\succsim$, defined on $\mathbb{R}^N$, satisfies Translation Scale Invariance if for all $x(N), y(N) \in \mathbb{R}^N$, we have $x(N) \succsim y(N)$ iff $x(N) + \alpha \succsim y(N) + \alpha$ for all $\alpha \in \mathbb{R}^N$. This axiom says that utility differences can be compared interpersonally. A comprehensive treatment of the literature on social choice with interpersonal utility comparisons can be found in Bossert and Weymark (2004).

We consider a natural extension of this axiom to the infinite horizon setting in a weaker form. The Partial Translation Scale Invariance axiom demands invariance with respect to changes in origin in comparing utility profiles which are identical after some finite generation. This axiom is weaker than the extension of the standard unit comparability axiom to the infinite dimensional case. In addition to invariance with respect to origin shifts, the unit comparability axiom requires invariance with respect to uniform scale changes as well. See, for instance, Sen (1977) and d'Aspremont and Gevers (1977). Proposition 3 below is Theorem 1 in Basu and Mitra (2003). It completely characterizes the $\succsim_U$ welfare relation in terms of Finite Anonymity, Strong Pareto and Partial Translation Scale Invariance.

Proposition 3 (Basu and Mitra (2003), Theorem 1) *A SWR $\succsim$ satisfies Anonymity, Strong Pareto and Partial Translation Scale Invariance iff $\succsim_U$ is a subrelation of $\succsim$.*

The characterization in Proposition 3 can now be used to obtain a characterization of the $\mathcal{Q}$-Utilitarian relation.

Theorem 2 *If $\mathcal{Q}$ is a group of cyclic permutations with $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{S}$, then a SWR $\succsim$ on X satisfies $\mathcal{Q}$-Anonymity, Strong Pareto and Partial Translation Scale Invariance iff $\succsim_{QU}$ is a subrelation of $\succsim$.*

The characterization result can be strengthened further. We can replicate the analysis presented in Theorem 1 to show that $\mathcal{Q}$-Anonymity, Strong Pareto and Partial Translation Scale Invariance generate the SWR $\succsim_{QU}$.

We denote the set of SWRs on X satisfying $\mathcal{Q}$ -Anonymity, Strong Pareto and Partial Translation Scale Invariance by Ξ'. This set is non-empty as $\succsim_{QU} \in \Xi'$. Consider the following binary relation on X: For all $x, y \in X$

$$x \succsim' y \text{ iff } x \succsim y \quad \text{for all } \succsim \in \Xi'.$$

We can now prove

Theorem 3 *If $\mathcal{Q}$ is a group of cyclic permutations with $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{P}$, then $\succsim'$ is a SWR on X satisfying $\mathcal{Q}$-Anonymity, Strong Pareto and Partial Translation Scale Invariance. Moreover, $\succsim' \equiv \succsim_{QU}$.*

The details of the proof of Theorem 3 are omitted for the sake of brevity.

5 Comparison with the overtaking criterion and Basu–Mitra utilitarianism

In this section, we compare the rankings of the $\mathcal{Q}$-utilitarian relation with the overtaking relation and the Basu–Mitra utilitarian relation. We will provide a class of examples for which it is argued that the rankings of the $\mathcal{Q}$-Utilitarian SWR are far more acceptable than those of the overtaking relation or the Basu–Mitra utilitarian relation.

We first introduce the overtaking relation:

$$\begin{gathered} x \succsim_O y \text{ if and only if} \\ \exists \bar{N} \in \mathbb{N} \text{ such that } I(x(N)) \geq I(y(N)) \quad \text{for all } N \geq \bar{N}. \end{gathered} \tag{5}$$

This version of the overtaking criterion satisfies the following property:

$$\begin{gathered} \text{For all } x, y \in X \text{ and all } \bar{N} \in \mathbb{N} \text{ such that } I(x(N)) \geq I(y(N)) \\ \text{for all } N \geq \bar{N}, \text{ if } I(x(N)) > I(y(N)) \text{ for all } N \text{ in a subsequence of} \\ \{N' \in \mathbb{N} : N' \geq \bar{N}\}, \text{ then } x \succ_O y. \end{gathered} \tag{6}$$

We now provide two examples to illustrate the relative merits of the $\mathcal{Q}$-Utilitarian SWR.

Example 1 Consider two utility streams:

$$\begin{aligned} x &= (1\ 0\ 1\ 0\ 1\ 0\ \ldots) \\ y &= (0\ 1\ 0\ 1\ 0\ 1\ \ldots). \end{aligned} \tag{7}$$

Observe that x cannot be obtained from y (nor y from x) by applying a finite permutation. However, it has been suggested by Van Liedekerke and Lauwers (1997, p. 162) that x should be declared indifferent to y. We will compare the ranking of x and y made by the overtaking and the $\mathcal{Q}$-Utilitarian relations. Note that in the pair defined in (9), for all $N \geq 1$, $I(x(N)) \geq I(y(N))$ and for the subsequence of odd integers we obtain, $I(x(N)) > I(y(N))$. Using (6), we get that $x \succ_O y$. As the transformation that generates y from x (or x from y) is an infinite permutation,

the Basu-Mitra utilitarian relation declares x and y as non-comparable. To rank the profiles x and y using the $\mathcal{Q}$-Utilitarian relation, let us consider the following permutation:

$$\begin{cases} \pi(n) = n+1 & \text{if } n \text{ is odd} \\ \pi(n) = n-1 & \text{if } n \text{ is even.} \end{cases} \tag{8}$$

The infinite permutation matrix corresponding to π can be obtained from the equation $\pi = Pa$, where $a = (1, 2, 3, \ldots)$. It is easy to check that $P(2n)$ is a finite dimensional permutation for each $n \in \mathbb{N}$. This shows that $P \in \mathcal{Q} \subseteq \mathcal{S}$. Now, $Px = y$, and the $\mathcal{Q}$-Utilitarian relation satisfies the axiom of $\mathcal{Q}$-Anonymity, so $x \sim_{QU} Px = y$. □

Example 2 Consider the following utility streams:

$$u = (2/3\ 1/3\ \ 1\ \ 2/4\ 4/9\ 2/9\ \ldots)$$
$$v = (1/4\ 1/2\ 1/4\ 3/4\ 1/9\ 3/9\ \ldots).$$

One can generate the sequence u in the following way: $u_1 = 2/3$, $u_2 = 1/3$ and for all $n \geq 3$,

$$u_n = \begin{cases} \frac{2}{(\frac{n}{2})^2} & \text{if } n \text{ is even} \\ \frac{4}{(\frac{n+1}{2})^2} & \text{if } n \text{ is odd.} \end{cases}$$

Similarly, $v_1 = 1/4$, $v_2 = 1/2$ and for all $n \geq 3$,

$$v_n = \begin{cases} \frac{3}{(\frac{n}{2})^2} & \text{if } n \text{ is even} \\ \frac{1}{(\frac{n+1}{2})^2} & \text{if } n \text{ is odd.} \end{cases}$$

Clearly, u and v are non-comparable according to Basu–Mitra utilitarianism, since in the odd periods the u entry exceeds the v entry, and in the even periods the v entry exceeds the u entry. Since $\mathcal{Q} \subseteq \mathcal{S}$ and $\mathcal{S}$ allows the infinite permutation of 1 with 2, 3 with 4, and so on (as shown in the previous example), then under such a permutation Q, we have:

$$Qv = (1/2\ 1/4\ 3/4\ 1/4\ 3/9\ 1/9\ \ldots).$$

Thus, $u > Qv$ and hence, $x \succ_{QU} Qv$. Since $Qv \sim_{QU} v$, we also have $x \succ_{QU} y$ by the transitivity of the binary relation $\succsim_{QU}$. □

There is a pattern to the profiles discussed in Examples 1 and 2. In both cases one profile Pareto dominates the other after being transformed by an infinite permutation matrix necessarily different from the class of finite permutation matrices.

We now discuss a potential drawback of the $\mathcal{Q}$-Utilitarian relation. As we are dealing with utilitarian relations, at least when comparing summable utility streams, we should declare the stream that sums to a relatively larger number as being ranked better. This is not always the case with either the Basu–Mitra utilitarian relation or the $\mathcal{Q}$ -Utilitarian relation. Example 3 presents an example in the summable class of profiles in which the $\mathcal{Q}$-Utilitarian fails to compare them.

Example 3 Consider the streams x and y defined as follows:

$$x = (1\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2^3}\ \tfrac{1}{2^3}\ \ldots\ )$$
$$y = (1\ 1\ \tfrac{1}{2^2}\ \tfrac{1}{2^2}\ \tfrac{1}{2^4}\ \ldots\ ).$$

The patterns are repeated in x and y. For example, each two generation block (starting with an even generation) in x repeats the reciprocal of an odd power of 2. Observe that both the streams are summable and that $I(x) = 7/3 < 8/3 = I(y)$. Since Pareto dominance of x over y or y over x beyond some N never takes place, x and y are non-comparable according to the Basu-Mitra utilitarian relation. Moreover, there is no infinite permutation $P \in \mathcal{Q}$ such that Py Pareto dominates x. Suppose, on the contrary, there exists an infinite permutation matrix $P \in \mathcal{Q}$ such that $Py \succ_U x$. Since $P \in \mathcal{Q} \subseteq \mathcal{S}$, there exists $\hat{N} \in \mathbb{N}$ such that $P(\hat{N})$ is a finite dimensional permutation matrix. From the definitions of x and y it follows that there exists $\bar{n} > \hat{N}$ for which it must be true that $x_{\bar{n}} > y_n$ for all $n > \hat{N}$. This shows that $Py \succ_U x$ cannot be true, as it is impossible to achieve Pareto dominance after some finite generation even with infinite permutation matrices in the class $\mathcal{Q}$. So the $\mathcal{Q}$-Utilitarian relation also declares x and y to be non-comparable. □

Whether one can define a relation in the spirit of utilitarianism that incorporates both classes of examples (ones that follow the pattern in Examples 1 and 2 and the comparison of summable sequences as in Example 3) and can be characterized without postulating any form of continuity is not known at present. This question is left for future research.

Appendix[6]

We first prove a lemma which is used to prove Proposition 2.

Lemma 2 *If $\succsim$ is a SWR on X satisfying $\mathcal{Q}$-Anonymity and Strong Pareto, then $\succsim_{QS}$ is a subrelation of $\succsim$.*

Proof Let $\succsim$ be a SWR on X satisfying $\mathcal{Q}$-Anonymity and Strong Pareto. We will show that $x \succ_{QS} y \Rightarrow x \succ y$ and $x \sim_{QS} y \Rightarrow x \sim y$. Let $x \succ_{QS} y$. Then (1) there is some $P \in \mathcal{Q}$ such that $Px \geq y$ and (2) for every $Q \in \mathcal{Q}$ it is the case that $\neg(Qy \geq x)$. Let us denote Px by z. Since $\succsim$ satisfies $\mathcal{Q}$-Anonymity and $z \in X$, we get $z \sim x$. We now claim that $z \neq y$. Suppose it is true that $z = y$. Then from the definition of z, we get $Px = y$. On pre-multiplying both sides of this equation by the unique inverse of P, say P', we get $x = P'y$. This violates condition (2). So, it must be that $z \neq y$. In view of (1), we obtain that $z > y$. Since $\succsim$ satisfies Strong Pareto, $z \succ y$. Now $x \succ y$ follows from the $\mathcal{Q}$ -Anonymity axiom and transitivity of the binary relation $\succsim$.

Let $x \sim_{QS} y$. Then (a) there exist $P, Q \in \mathcal{Q}$ such that $Px \geq y$ and $Qy \geq x$. This implies

$$QPx \geq Qy \geq x.$$

[6] The method of proof we adopt here was suggested, in part, by an anonymous referee of this journal.

Since $\mathcal{Q}$ is a group of cyclic permutations and $P, Q \in \mathcal{Q}$, it must be true that $QP \in \mathcal{Q}$ and by Lemma 1 it is the case that $\neg(QPx > x)$. Thus, $QPx = Qy$ holds, and pre-multiplying by Q' (the unique inverse of Q) we get $Px = y$. By the $\mathcal{Q}$ -Anonymity axiom, we obtain $x \sim Px = y$ as was required. □

Proof of Proposition 2 Using Proposition 1 and Lemma 2 above, the proposition is easily seen to hold. □

Two important characterization results are established in Lemmas 3 and 4.

Lemma 3 *For any $x, y \in X$,*

$$x \succ_U y \text{ iff } \exists\, N \in \mathbb{N} \text{ such that } (I(x(N)), x[N]) > (I(y(N)), y[N]) \tag{9}$$

and

$$x \sim_U y \text{ iff } \exists\, N \in \mathbb{N} \text{ such that } (I(x(N)), x[N]) = (I(y(N)), y[N]). \tag{10}$$

Proof We prove the equivalence in (9). The equivalence in (10) can be proved easily using the proof of (9) and the definition of $\succsim_U$. We omit the details for the sake of brevity. It follows from the definition of $\succsim_U$ that $x \succ_U y$ is equivalent to

$$\exists\, N \in \mathbb{N} \text{ such that } (I(x(N)), x[N]) \geq (I(y(N)), y[N]) \tag{11}$$

and

$$\nexists\, N' \in \mathbb{N} \text{ satisfying} (I(y(N')), y[N']) \geq (I(x(N')), x[N']). \tag{12}$$

(only if part) Assume $x \succ_U y$. Suppose there exists $N \in \mathbb{N}$ such that $(I(x(N)), x[N]) = (I(y(N)), y[N])$. This contradicts (12). By (11), there exists $N \in \mathbb{N}$ such that $(I(x(N)), x[N]) > (I(y(N)), y[N])$.

(if part) Assume that there exists $N \in \mathbb{N}$ such that $(I(x(N)), x[N]) > (I(y(N)), y[N])$. So (11) must hold. It also implies that there is some $N^* \geq N$ such that for all $N' \geq N^*$, we have $I(x(N')) > I(y(N'))$ and $x[N'] \geq y[N']$. From this we conclude that (12) must hold. Since (11) and (12) both hold $x \succ_U y$ is established. □

Lemma 4 *If $\mathcal{Q}$ is a group of cyclic permutations with $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{S}$ and if $P \in \mathcal{Q}$, then for any $x, y \in X$, $x \succsim_U y$ iff $Px \succsim_U Py$.*

Proof (only if part) Assume $x \succsim_U y$ and $P \in \mathcal{Q}$. Since $\mathcal{Q} \subseteq \mathcal{S}$, there is some $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $P(kn)$ is a finite dimensional permutation matrix. Let us denote $Px = x'$ and $Py = y'$. As $x \succsim_U y$, there is some $\hat{n} \in \mathbb{N}$ such that for $N = \hat{n}k$,

$$(I(x(N)), x[N]) \geq (I(y(N)), y[N]). \tag{13}$$

From (13), we know that the infinite utility stream $x[N] \geq y[N]$. Now, since $P \in \mathcal{Q} \subseteq \mathcal{S}$, P_N is an infinite permutation matrix in its own right. Using (13), this gives

$$P_N x[N] \geq P_N y[N].$$

From the choice of N, $P(N)$ is a finite dimensional permutation matrix. This implies that $I(x(N)) = I(x'(N))$ and $I(y(N)) = I(y'(N))$. Using (13), we now obtain $I(x'(N)) \geq I(y'(N))$, which proves that $Px \succsim_U Py$.

(if part) Assume $Px \succsim_U Py$ for some $P \in \mathcal{Q}$ and $x, y \in X$. Observe that $Px, Py \in X$. Let us denote by P' the unique inverse of P. Note that $P' \in \mathcal{Q}$ (because $\mathcal{Q}$ is a group) and $P'(Px) = x$ and $P'(Py) = y$. Applying the "only if" part of the proof to the relation $Px \succsim_U Py$ we get $P'(Px) \succsim_U P'(Py)$. This is equivalent to $x \succsim_U y$. □

We state a corollary to Lemma 4. The proof is a straightforward application of Lemma 4 and is omitted.

Corollary 1 *If $\mathcal{Q}$ is a group of cyclic permutations with $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{S}$ and if $P \in \mathcal{Q}$, then for any $x, y \in X$,* (a) *$x \succ_U y$ iff $Px \succ_U Py$ and* (b) *$x \sim_U y$ iff $Px \sim_U Py$.*

We now need to show that $\succsim_{QU}$ is a SWR. This is proved in Lemma 5.

Lemma 5 *If $\mathcal{Q}$ is a group of cyclic permutations with $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{S}$, then $\succsim_{QU}$ is a SWR.*

Proof Reflexivity of $\succsim_{QU}$ follows from the fact that $I \in \mathcal{Q}$ and $\succsim_U$ is reflexive. To check transitivity, let $x \succsim_{QU} y$ and $y \succsim_{QU} z$ for some $x, y, z \in X$. Then there is some $P, Q \in \mathcal{Q}$ such that $Px \succsim_U y$ and $Qy \succsim_U z$. Using Lemma 4 and the fact that $\mathcal{Q}$ is a group (this ensures that $QP \in \mathcal{Q}$), we get $QPx \succsim_U Qy \succsim_U z$. Transitivity of $\succsim_U$ implies $QPx \succsim_U z$. From the definition in (2), it follows that $x \succsim_{QU} z$. □

The next lemma establishes a characterization of the strict relation and the indifference relation corresponding to $\succsim_{QU}$.

Lemma 6 *If $\mathcal{Q}$ is a group of cyclic permutations with $\mathcal{F} \subseteq \mathcal{Q} \subseteq \mathcal{S}$, then*

$$x \succ_{QU} y \text{ iff } \exists\, P \in \mathcal{Q} \text{ such that } Px \succ_U y \tag{14}$$

and

$$x \sim_{QU} y \text{ iff } \exists\, P \in \mathcal{Q} \text{ such that } Px \sim_U y. \tag{15}$$

Moreover, $\succsim_U$ is a subrelation of $\succsim_{QU}$.

Proof [only if part of (14)]. Assume $x \succ_{QU} y$. It follows from the definition of $\succsim_{QU}$ that $x \succ_{QU} y$ is equivalent to

$$\exists P \in \mathcal{Q} \text{ such that } Px \succsim_U y \tag{16}$$

and

$$\forall Q \in \mathcal{Q}, \neg(Qy \succsim_U x). \tag{17}$$

So there exists $P \in \mathcal{Q}$ such that $Px \succsim_U y$. Let P' denote the unique inverse of P in $\mathcal{Q}$. By Lemma 4, $x = P'(Px) \succsim_U P'y$. Since (17) is true for all $Q \in \mathcal{Q}$, by substituting $Q = P'$ in (17) we have $\neg(P'y \succsim_U x)$. This together with $x \succsim_U P'y$ implies $x \succ_U P'y$. Using Corollary 1, we get $Px \succ_U y$.

[if part of (14)] Assume that there exists $P \in \mathcal{Q}$ such that $Px \succ_U y$. From the definition of $\succsim_{QU}$, $x \succsim_{QU} y$. Let us suppose, contrary to what needs to be shown, $y \succsim_{QU} x$. So there exists a $Q \in \mathcal{Q}$ such that $Qy \succsim_U x$. By Lemma 4, $PQy \succsim_U Px$. By transitivity of $\succsim_U$ and $Px \succ_U y$, $PQy \succ_U y$. Let R denote the product $PQ \in \mathcal{Q}$ and z the utility stream Ry. By Lemma 3, there exists $N \in \mathbb{N}$ such that $(I(z(N)), z[N]) > (I(y(N)), y[N])$. So there is some $N^* \geq N$ such that for all $N' \geq N^*$, $I(z(N')) > I(y(N'))$. Since $R \in Q \in \mathcal{Q} \subseteq \mathcal{S}$, there is some $\hat{N} \geq N^*$ such that $R(\hat{N})$ is a finite dimensional permutation matrix. This implies $I(z(\hat{N})) = I(y(\hat{N}))$. This leads to a contradiction to the fact that for all $N' \geq N^*$, $I(z(N')) > I(y(N'))$. So we must have $\neg(y \succsim_{QU} x)$, which proves that $x \succ_{QU} y$.

[only if part of (15)] Assume $x \sim_{QU} y$. It follows from the definition of $\succsim_{QU}$ that $x \sim_{QU} y$ is equivalent to

$$\exists P \in \mathcal{Q} \text{ such that } Px \succsim_U y \tag{18}$$

and

$$\exists Q \in \mathcal{Q} \text{ such that } Qy \succsim_U x. \tag{19}$$

We first prove that $y \succsim_U Px$. Suppose on the contrary, $\neg(y \succsim_U Px)$. Using (18), $Px \succ_U y$. In view of (14), we would have $x \succ_{QU} y$, contradicting the assumption that $x \sim_{QU} y$. So we have $Px \sim_U y$.

[if part of (15)] Assume $\exists\ P \in \mathcal{Q}$ such that $Px \sim_U y$. Then from the definition of $\sim_U$, we have (18) and (19) are both satisfied (with Q replaced by P) implying $x \sim_{QU} y$ as was required.

Since $I \in \mathcal{Q}$, $\succsim_U$ is a subrelation of $\succsim_{QU}$. □

Proof of Theorem 1 It is easy to check that $\succsim^*$ is a SWR on X. We need to show (a) $\succsim_{QS}$ is a subrelation of $\succsim^*$ and (b) $\succsim^*$ is a subrelation of $\succsim_{QS}$. It is also easy to show that the SWR $\succsim^*$ satisfies $\mathcal{Q}$-Anonymity and Strong Pareto. Using Proposition 2, this implies that $\succsim_{QS}$ is a subrelation of $\succsim^*$. We will now show that $\succsim^*$ is a subrelation of $\succsim_{QS}$.

Suppose $x \succ^* y$. We want to show that $x \succ_{QS} y$. From the definition of $\succsim^*$, $x \succsim_{QS} y$, since $\succsim_{QS} \in \Xi$. It has to be shown that $\neg(y \succsim_{QS} x)$. Suppose on the contrary, $y \succsim_{QS} x$. Then by Proposition 2 and the fact that $\succsim^* \in \Xi$, we must have $y \succsim^* x$. This is a contradiction to $x \succ^* y$. This implies that $x \succsim_{QS} y$ and $\neg(y \succsim_{QS} x)$. Hence, $x \succ_{QS} y$.

We will now show that $x \sim^* y$ implies $x \sim_{QS} y$. Suppose $x \sim^* y$. From the definition of $\succsim^*$, it follows that $x \succsim_{QS} y$ and $y \succsim_{QS} x$, since $\succsim_{QS} \in \Xi$. This proves $x \sim_{QS} y$. So $\succsim$ 651$*$ is a subrelation of $\succsim_{QS}$. □

Proof of Theorem 2 (A SWR $\succsim$ satisfies $\mathcal{Q}$-Anonymity, Strong Pareto and Partial Translation Scale Invariance only if $\succsim_{QU}$ is a subrelation of $\succsim$.) Assume that a SWR $\succsim$ on X satisfies $\mathcal{Q}$-Anonymity, Strong Pareto and Partial Translation Scale Invariance. Recall that the inverse of a matrix $P \in \mathcal{Q}$ is denoted by P'.

Let $x \succ_{QU} y$. By Lemma 6, there exists $P \in \mathcal{Q}$ such that $Px \succ_U y$. By Proposition 3, $Px \succ y$. Since $\succsim$ satisfies $\mathcal{Q}$-Anonymity, $x = P'Px \sim Px \succ y$ and by transitivity, $x \succ y$.

Let $x \sim_{QU} y$. By Lemma 6, there exists $P \in \mathcal{Q}$ such that $Px \sim_U y$. By Proposition 3, $Px \sim y$. Since $\succsim$ satisfies $\mathcal{Q}$-Anonymity, $x = P'Px \sim Px \sim y$ and by transitivity, $x \sim y$.

This shows that $\succsim_{QU}$ is a subrelation of $\succsim$.

(A SWR $\succsim$ satisfies $\mathcal{Q}$-Anonymity, Strong Pareto and Partial Translation Scale Invariance if $\succsim_{QU}$ is a subrelation of $\succsim$.) Assume that $\succsim_{QU}$ is a subrelation of $\succsim$.

($\mathcal{Q}$-Anonymity) Let $Q \in \mathcal{Q}$. We have to show that $Qx \sim x$. $Q'Qx = x \sim_U x$ since $\succsim_U$ is reflexive. By Lemma 6, $Qx \sim_{QU} x$. Since $\succsim_{QU}$ is a subrelation of $\succsim$, $Qx \sim_{QU} x$ implies $Qx \sim x$.

(Strong Pareto) Let $x > y$. We have to show $x \succ y$. Since $\succsim_U$ satisfies Strong Pareto, $x \succ_U y$. This implies, using Lemma 6, $x \succ_{QU} y$. Since $\succsim_{QU}$ is a subrelation of $\succsim$, $x \succ y$.

(Partial Translation Scale Invariance) Assume $(x(N), x[N]) \succsim (y(N), x[N])$, $\alpha \in \mathbb{R}^{\mathbb{N}}$, $(x(N), x[N]) + \alpha \in X$ and $(y(N), x[N]) + \alpha \in X$. We have to show that $(x(N), x[N]) + \alpha \succsim (y(N), x[N]) + \alpha$. We now claim that $I(x(N)) \geq I(y(N))$. Suppose on the contrary $I(y(N)) > I(x(N))$. This implies from the definition of $\succsim_U$, $(y(N), x[N]) \succ_U (x(N), x[N])$. By Lemma 6, $\succsim_U$ is a subrelation of $\succsim_{QU}$, so $(y(N), x[N]) \succ_{QU} (x(N), x[N])$ holds. By assumption $\succsim_{QU}$ is a subrelation of $\succsim$. This implies $(y(N), x[N]) \succ (x(N), x[N])$ which contradictions our initial assumption that $(x(N), x[N]) \succsim (y(N), x[N])$. This shows that $I(x(N)) \geq I(y(N))$. From this it follows that $(x(N), x[N]) + \alpha \succsim_U (y(N), x[N]) + \alpha$. By Lemma 6, $\succsim_U$ is a subrelation of $\succsim_{QU}$, so $(x(N), x[N]) + \alpha \succsim_{QU} (y(N), x[N]) + \alpha$. By assumption, $\succsim_{QU}$ is a subrelation of $\succsim$. This implies that $(x(N), x[N]) + \alpha \succsim (y(N), x[N]) + \alpha$. □

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