On the Representation of Preference Orders on Sequence Spaces

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Abstract

A set of sufficient conditions for representability of preference orders on real sequence spaces is analyzed. In particular, monotonicity and continuity of the order is not assumed. Two applications are worked out to demonstrate how such a result might be useful.

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1 Introduction

In this note we study a set of sufficient conditions guaranteeing representability of complete binary relations on real sequence spaces. In a recent paper, Mitra and Ozbek (2013) (hence forth Mitra-Ozbek) showed that representability of monotone orders (where a weak dominance of one sequence over the other translates to a weak preference, a formal definition is section 2.2) can be guaranteed if the upper and lower contour sets defined along the diagonal (that is, set of all constant profiles being no worse than and no better than a given element) are both closed (this is formally stated in section 2.2). This condition Scalar Continuity (see section 2.2 for a formal definition) is sufficient for the existence of a utility function in $\mathbb{R}^n$ for strongly monotone[1] preferences. This route of existence of a utility function is taken in the text book treatment of this subject matter even though the stronger continuity assumption[2] is made (see Mas-Colell, Whinston and Green (1995) and Jehle and Reny (2011) for example). The Mitra-Ozbek result (and results reported here) points to a more general applicability of the scalar continuity condition and of the proof technique essentially due to Wold (1943). As a departure from the Mitra-Ozbek result we emphasize a different monotonicity condition. Our condition (Diagonal Pareto) states that if a diagonal element of the sequence space (elements of the form $(\lambda, \lambda, ..., \lambda)$) dominates another diagonal element then the order declares the dominant diagonal element as being strictly preferred.

In relation to the larger literature on representability the main result of this paper (Theorem 1, section 3) can be seen as a contribution to the theory of representation that identifies easy-to-check conditions beyond order denseness. In this regard the analysis of the current paper can be seen as being along the lines of Wold (1943), Rader (1963), Eilenberg (1941) among others. It is worth mentioning that there is no escaping order denseness, which is well known to characterize the class of representable orders on arbitrary sets (see Debrue (1964) and Jaffray (1975) among others).

As an application of our result (section 4.1) we demonstrate that a class of inequity averse preferences that fails to be monotone (but satisfies diagonal Pareto) is representable under our assumptions but its representability does not follow from the Mitra-Ozbek result. As a second application (section 4.2) we show that an important class of benevolent preferences also satisfy diagonal Pareto and hence, our main result provides an alternative set of sufficient conditions that imply that this class of preferences is representable. A summary of our findings conclude the paper (section 5).

[1]If each $x_i \geq y_i$ and the inequality is strict for some i then a strongly monotone preference will declare x to be strictly preferred to y.

[2]The upper and lower contour sets for each element is assumed closed in the natural topology of the space.

2 Preliminaries

2.1 Notation and Definitions

Let X be a (sequence) space of the form Y^M where Y is a non-degenerate interval[3] of $\mathbb{R}$ and $M \in \mathbb{N}$. We can view elements of X as payoffs, utility profiles (each element of the M-vector pertaining to the utility of an individual in society) as scenarios covered under this setting.

The object of enquiry is the representation of complete, transitive binary relations $\succsim$ (called an preference order) defined on X. The following notation defined on elements of X will be maintained throughout the analysis: for $x, y \in X$ we say $x \geq y$ iff $x_i \geq y_i$ for all $i \in M$; $x > y$ iff $x \geq y$ iff $x \geq y$ and $x \neq y$; $x >> y$ iff $x_i > y_i$ for all $i \in M$.

An order $\succsim$ on X is *representable* if there exists a real-valued function $u : X \to \mathbb{R}$ such that

$$x \succsim y \text{ iff } u(x) \geq u(y). \tag{R}$$

2.2 Monotonicity Conditions and Continuity

This section documents the conditions that are needed for our representation result. For ease of exposition we write e for the constant M-vector of all $1's$. Two sets are of particular interest (as in Mitra-Ozbek): for each $x \in X$ define the sets $A(x) = \{\lambda \in Y : \lambda e \succsim x\}$ and $B(x) = \{\lambda \in Y : x \succsim \lambda e\}$. An order $\succsim$ on X satisfies

(DP) Diagonal Pareto: if for any $\lambda, \mu \in Y$ whenever $\lambda > \mu$ the relation $\lambda e \succ \mu e$ holds

(SC) Scalar Continuity: if for each $x \in X$, the sets $A(x)$ and $B(x)$ are closed subsets in Y

(NE) Non-emptiness: if for each $x \in X$, the set $A(x)$ and $B(x)$ are both non-empty.

If any form of monotonicity makes sense in a particular application we can be assured that diagonal Pareto will not be out of place. In economic settings where Pareto conditions (of which **DP** is one) serve as measure of efficiency it often comes in conflict with equity. This is no longer a concern when we restrict attention to the diagonal, which is equitable by virtue of being a constant profile. For easy reference we state the monotonicity condition from Mitra-Ozbek and their main representation result (Proposition 2 in Mitra and Ozbek (2013):

(M) Monotonicity: for $x, y \in X$ if $x \geq y$, then $x \succsim y$

(MOR) *Mitra-Ozbek Representation: If $\succsim$ satisfies condition M and SC, then $\succsim$ is representable.*

[3]This means that Y is any subset of $\mathbb{R}$ taking one of the following forms: (a) $(a, b]$ with $-\infty \leq a < b < \infty$ (b) (a, b) with $-\infty \leq a < b \leq \infty$ (c) $[a, b]$ with $-\infty < a < b < \infty$ and (d) $[a, b)$ with $-\infty < a < b \leq \infty$.

3 Representation Theorem

In this section we report the main result of the paper. It is shown that the conditions **NE, SC** and **DP** are sufficient to guarantee representation in sequence spaces.

Theorem 1: *Suppose* $X = Y^M$ for some non-degenerate interval in $\mathbb{R}$ and $M \in \mathbb{N}$. *If* $\succsim$ *on* X *satisfies NE, SC and DP, then* $\succsim$ *is representable.*

Proof: Under **NE** and **SC** the sets $A(x)$ and $B(x)$ are non-empty and closed subsets of Y. Moreover since $\succsim$ is complete $A(x) \cup B(x) = Y$, and as Y is connected (Proposition 12, p. 183 Royden (1988)) it must be that $A(x) \cap B(x)$ must be non-empty. Now **DP** implies that there is a unique element in $A(x) \cap B(x)$ for each $x \in X$. For if $\lambda, \mu \in A(x) \cap B(x)$ with $\lambda > \mu$ for some $x \in X$, then by **DP** $x \sim \lambda e \succ \mu e \sim x$, a violation of reflexivity. Denote the unique element of $A(x) \cap B(x)$ by $u(x)$.

Suppose $x, y \in X$ and $x \succ y$. We will show that $u(x) > u(y)$. It follows from the definition of u that $u(x)e \sim x \succ y \sim u(y)e$, and by transitivity we get $u(x)e \succ u(y)e$. By **DP** we must have $u(x) > u(y)$ as was needed. When $x \sim y$ we have $u(x)e \sim x \sim y \sim u(y)e$ which implies using **DP**, $u(x) = u(y)$. ■

Remark 1:

(i) Method of Proof: The method of proof essentially uses the Wold technique (Wold (1943)). For each x this technique finds an element on the diagonal that is indifferent to x. For this method to work it is necessary that the sets $A(x)$ and $B(x)$ be non-empty for each x. Note that condition **M** is not directly comparable with **DP**, while **M** says that for $\lambda > \mu$ the relation $\lambda e \succsim \mu e$ must hold, it is silent about whether the preference is strict. On the other hand **DP** has no say over profiles off the diagonal. This has the consequence that Theorem1 (which uses **DP**) cannot be directly inferred from **MOR** nor does Theorem 1 imply **MOR**.

(ii) Continuous Representation: A natural step beyond representability is to ask whether one can always find a continuous function that represents $\succsim$ satisfying **NE, SC** and **DP**. In general, we observe that the function representing preferences satisfying conditions **NE**, **SC** and **DP** need not be continuous. This fact is illustrated through a simple example in $X = \mathbb{R}^2$. Let $x \succsim y$ iff $u(x) \geq u(y)$ where $u : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$u(x_1, x_2) = \begin{cases} x_1 & \text{for } x_1 < x_2 \\ x_1 + x_2 & \text{for } x_1 \geq x_2. \end{cases}$$

For $x \in X$ with $x_1 < x_2$ we have $x \sim (x_1/2)e$ and for $x \in X$ with $x_1 \geq x_2$ it is easy to see that $((x_1 + x_2)/2)e \sim x$. Hence $\succsim$ satisfies **NE**. For any pair $\lambda > \mu$, we have $u(\lambda e) = 2\lambda > 2\mu = u(\mu e)$ hence $\lambda e \succ \mu e$ implying **DP** is also satisfied. To verify **SC**, take $x \in \mathbb{R}^2$ such that $x_1 < x_2$ and observe that $A(x) = [x_1/2, \infty)$ and $B(x) = (-\infty, x_1/2]$ where both sets are closed in $\mathbb{R}$. Similarly for x in $\mathbb{R}^2$ such that $x_1 \geq x_2$ we must have $A(x) = [(x_1+x_2)/2, \infty)$ and $B(x) = (-\infty, (x_1+x_2)/2]$ again both sets are closed in $\mathbb{R}$. Hence **SC** is satisfied as well.

However u is *not* continuous on $\mathbb{R}^2$. The sequence

$$x_n = (2 - (1/n), 2 + (1/n)) \text{ for } n \geq 1 \tag{1}$$

converges to $(2,2)$ but $\lim_{n\to\infty} u(x_n) = \lim_{n\to\infty}(2-(1/n)) = 2 < 4 = u(2,2)$. So $\succsim$ is not continuous (in the sense that the upper and lower contour sets associated with each x are closed in the usual topology of $\mathbb{R}^2$) and hence there is *no* continuous function that represents $\succsim$. For the sake of completeness we provide a short argument. Assume on the contrary the existence of continuous function $v : X \to \mathbb{R}$ that represents $\succsim$. Consider the sequence $\{x_n\}$ for $n \geq 1$ as defined in (1) and note that (i) the sequence x_n converges to $(2,2)$ and (ii) since v represents $\succsim$ we must have $v(x_n) < v(2,5/2)$, as $(2-(1/n), 2+(1/n)) \prec (2,5/2)$. But the assumed continuity of v also implies that $v(2,2) \leq v(2,5/2)$ a contradiction to the fact that v represents $\succsim$, as representability would imply $v(2,2) > v(2,5/2)$ since $(2,2) \succ (2,2/5)$.

4 Two Examples

4.1 Inequity Averse Preferences

In this section we study the representation problem of a class of preferences that is sensitive to inequity[4]. We seek to model a planners preference over pairs in $\mathbb{R}^M$ where $M \geq 2$ is the number of individuals in society. Assume that the planner has some influence on the income generating process in the economy (for example, can credibly implement a tax-subsidy policy explicitly affecting individual incomes) and preferences over the set of possible deviations from the current income levels of individuals in society. With this interpretation in mind, the reference point (status quo) is treated as the zero vector and each x measures the respective loss or gain from the implemented policy as deviation from the status quo.

Formally preferences are defined on $X = \mathbb{R}^M$ with each component x_i signifying a loss (if $x_i < 0$) or a gain (if $x_i > 0$) in individual i's income. Additionally it is assumed that the planner is inequity averse and this aversion is explicitly modelled with reference to some inequality measure[5], a function $I : X \to \mathbb{R}_+$ satisfying:

(N) Normalization: For every $\lambda \in \mathbb{R}$, the inequality measure $I(\lambda e) = 0$ and $I(x) > 0$ for all non-constant x.

Normalization says that any deviation from the perfectly equitable profile (i.e., profiles taking values on the diagonal) exhibits some degree of inequity.

(EWP) Equity Adjusted Weak Pareto: For $x, y \in X$ if $(x - I(x)e) >> (y - I(y)e)$, then $x \succ y$.

Condition **EWP** summarizes a conservative approach to declaring Pareto dominant streams as strictly preferred. This condition states that if each x_i dominates y_i net of the inequity in their respective profiles, then we should declare x better than y. This is in contrast to the condition weak Pareto,

[4]Inequity Averse preferences are widely observed in experimental games, see Fehr and Schmidt (1999) and a general axiomatic characterization of individual inequity averse preferences by Neilson (2006).

[5]The inequality measurement literature is extensive, see the seminal paper by Atkinson (1970) as an intuitive summary of the pertinent issues involved.

which would say $x >> y$ implies $x \succ y$. Condition **EWP** is stronger than both **DP** and **NE**. As a result of this, assuming **EWP** and **SC** suffices to obtain a representation. This is the content of Proposition 1.

Proposition 1: *Suppose $X = \mathbb{R}^M$ and that inequity in utility profiles is measured using an inequality measure $I : X \to \mathbb{R}_+$ satisfying N. If $\succsim$ is a preference order on X satisfying EWP and SC, then it is representable.*

Proof: It will be shown that all conditions needed to invoke Theorem 1 are satisfied. To show that **DP** holds, take $\lambda > \mu$ for some $\lambda, \mu \in \mathbb{R}$. We need to show the relation $\lambda e \succ \mu e$ is true. Observe that $\lambda e >> \mu e$ and by **N** we have $I(\lambda e) = I(\mu e) = 0$, so by **EWP** we get $\lambda e \succ \mu e$. Hence, $\succsim$ satisfies **DP**.

Let $x \in X$, we will show that $A(x)$ is non-empty. Denote $\max\{x_i : i = 1, ..., n\}$ by α and note that for $\epsilon > 0$ we must have $(\alpha + \epsilon)e >> x$ and $0 = I((\alpha + \epsilon)e) < I(x)$ (by **N**). Now **EWP** implies $(\alpha + \epsilon)e \succ x$ proving that $A(x)$ is non-empty. $B(x)$ is also non-empty; to see this denote by β the quantity $\min\{x_i - I(x) : i = 1, ..., n\}$, then for any $\epsilon > 0$ we must have $(x - I(x)e) >> (\beta - \epsilon)e$ and $I((\beta - \epsilon)e) = 0$ (by **N**). Now using **EWP** that $x \succ (\beta - \epsilon)e$. Hence, $B(x)$ is non-empty. We have shown that both **NE** and **DP** hold. Since **SC** is assumed, by Theorem 1 we can conclude that $\succsim$ is representable. ■

Let us work out a specific example in this class of preference and demonstrate the use of our result. Define $\succsim$ on X as $x \succsim y$ iff $u(x) \geq u(y)$ where $u : X \to \mathbb{R}$ is

$$u(z) = m(z) - I(z) \tag{2}$$

for all $z \in X$ and $m(z) = \min\{z_i : i = 1, ..., M\}$. It is easy to check that $\succsim$ satisfies **EWP**. If $x, y \in X$ and $(x - I(x)e) >> (y - I(y)e)$, then it follows immediately that $m(x) - I(x) > m(y) - I(y)$ holds, implying $x \succ y$. To see that **SC** holds note that for $x \in X$ we must have $A(x) = [m(x) - I(x), \infty)$ and $B(x) = (-\infty, m(x) - I(x)]$; as both sets are closed in $\mathbb{R}$ condition **SC** is verified.

Existing results (Mitra and Ozbek (2013), Segal and Sobel (2001) for example) with sufficient conditions involving monotonicity of the preference order cannot be used to claim representability of this class of preferences, since not all members of this class satisfies the basic monotonicity condition **M**. To see this, consider the preference given by (2) when $X = \mathbb{R}^M$ for $M \geq 2$ and $I(x_1, x_2, ..., x_M) = \sum_{i \neq j} |x_i - x_j|$. Clearly, I satisfies condition **N**. Let us compare the two profiles $(1/2)e$ and $(2, 1, 1, ..., 1)$. Note that $u((1/2)e) = (1/2) > 0 \geq 2 - M = 1 - (M - 1) = u(2, 1, 1, .., 1)$ implying $(1/2)e \succ (2, 1, 1, ..., 1)$ in violation of monotonicity condition **M**.

4.2 Benevolent Preferences

Consider preferences defined on $X = \mathbb{R}^{n+1}_{++}$ and a generic element from X will be written as (a, x) with $a \in \mathbb{R}_{++}$ and $x \in \mathbb{R}^n_{++}$. We will interpret the vector (a, x) as to how the subject whose income is a perceives being in the state of the world where the income profile of the rest of society is x. Given a preference order $\succsim$ (a complete and transitive) binary relation on X the preference relation

$(a, x) \succsim (b, y)$ is a shorthand for the subject's preference to be social state x with income a than be in the social state y with personal income b. The general class of preferences where an agent's assessment of his personal well being is dependent on how he views the profile of income of others (in society or peer group) are called *interdependent preferences.* Two distinctions are natural: (i) if an individual's overall well being is adversely affected by the well being of others in society (negatively interdependent preferences; Ok and Koçkesen (2000)) and (ii) if an agent own well being exhibits a positive interdependence towards "others" well being (benevolent or positively interdependent preferences).

In what follows we study a simple model of benevolent preferences and demonstrate that such preferences can always be represented using Theorem 1. The vector $(0, .., 0, 1, 0, ..., 0)$ (with the 1 appearing at position i) will be written as e^i and the vector $(1, ...1)$ in $\mathbb{R}^n_{++}$ will be written as e_n. Before formally stating the formal definition of benevolent preferences two natural properties of such preferences are stated.

(B1) For $a > b$ we must have $(a, x) \succ (b, x)$ for all $x \in \mathbb{R}^n$.

(B2) For any permutation $\pi : \{1, ..., n\} \rightarrow \{1, ..., n\}$ we must have $(a, x) \sim (a, \pi \cdot x)$ where $\pi \cdot x = (x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$.

Properties **B1** and **B2** are standard. **B1** states that comparing income across two identical social states an agent chooses the profile where his personal income is higher. **B2** states that beyond his personal income, an agents cares about the distribution of income not the identity of the individuals that are the recipients of the income.

We say that a preference order $\succsim$ on $X = \mathbb{R}^{n+1}_{++}$ is *benevolent or positively interdependent* if

$$(a, (x_1, x_2, ..., x_n)) \succsim (a, (y_1, y_2, ..., y_n))$$

whenever

$$(x_i, (a, x_{-i})) \succsim (y_i, (a, y_{-i})) \text{ for all } i = 1, ..., n. \tag{3}$$

and $(a, (x_1, x_2, ..., x_n)) \succ (a, (y_1, y_2, ..., y_n))$ if at least one of the preference relation in (3) is strict. Intuitively, condition (3) states that if every member of society prefers to being in social state x than in y, so does a benevolent agent whose utility is a in both states.

A preference order $\succsim$ on $X = \mathbb{R}^{n+1}_{++}$ exhibits *weak Paretian altruism* if for any a, b with $b > a$ the strict preference $(a, be_n) \succ (a, ae_n)$ holds.

Before we proceed let us briefly describe some features of this model of preferences. Firstly, this class of preferences is not new, our definition of benevolence is the dual of the definition of negatively interdependent preferences in Ok and Koçkesen (2000). Secondly, for this class of preferences it is *not* known whether

$$(a, x) \succ (a, y) \text{ when } x > y \text{ holds,} \tag{4}$$

neither are there examples of positively interdependent preferences satisfying **B1**, **B2** which violate (4). The dual of this fact is stated as an open problem for negatively interdependent preferences in Ok and Koçkesen (2000, p. 542 in the Remarks immediately following Proposition 1). We are not able to resolve the status of that question (or the analog of that question for positively interdependent preferences, namely condition (4)) but we can show that every positively interdependent preference order satisfying **B1**, **B2** must satisfy weak Paretian altruism (Lemma 1). In other words we can show (4) for constant sequences. Whether the assumptions made in this paper (or the stronger continuity assumption made in Ok and Koçkesen (2000)) resolve this open question is not known as of now.

Lemma 1: *If $\succsim$ is a benevolent preference order on $X = \mathbb{R}^{n+1}_{++}$ satisfying B1, B2, then it satisfies weak Paretian altruism.*

Proof: Let $b > a$, we have to show that $(a, be_n) \succ (a, ae_n)$. Note that (a, be_n) can be obtained from (a, ae_n) by replacing in last n terms (the $a's$) by $b's$ one at a time. Denote by x^k the profile in X defined by

$$x^k = \begin{cases} (a, ae_n) & \text{for } k = 0 \\ (a, (be_k, ae_{n-k})) & \text{for } k = 1, .., n-1 \\ (a, be_n) & k = n. \end{cases}$$

We will show that $x^1 \succ x^0$ and $x^k \succsim x^{k-1}$ for $k = 2, ..., n$. Let us show $x^1 \succ x^0$. Observe that by **B1** we must have $(b, ae_n) \succ (a, ae_n)$, since $b > a$. How does the individual endowed of this preference order perceive how "other" members of society view the profiles (b, ae_n) and (a, ae_n)? This is dictated by (3) and reduces to comparing $(a, (b, ae_{n-1}))$ and (a, ae_n). Note the position of b in the first profile is irrelevant (in the sense that for every other position b takes within (b, ae_{n-1}), the resultant profile will be indifferent to $(a, (b, ae_{n-1})))$ by **B2**. By the positive interdependence, since $(b, ae_n) \succ (a, ae_n)$ we must have $(a, (b, ae_{n-1})) \succ (a, ae_n)$. This establishes that $x^1 \succ x^0$.

Now we establish $x^{k+1} \succsim x^k$ for some k satisfying $1 < k < n$. This generic case suffices to establish the result and demonstrates the logic behind the main argument. Consider the two profiles $(b, (be_k, ae_{n-k}))$ and $(a, (be_k, ae_{n-k}))$. By **B1** we must have

$$(b, (be_k, ae_{n-k})) \succ (a, (be_k, ae_{n-k})). \tag{5}$$

Assume, contrary to what needs to be shown, that $x^{k+1} \prec x^k$ holds. Using (5) and transitivity ($x^{k+1} \prec x^k$ and $x^k \prec (b, (be_k, ae_{n-k}))$) we get

$$(a, (be_{k+1}, ae_{n-k-1})) \prec (b, (be_k, ae_{n-k})). \tag{6}$$

The comparisons dictated by the definition of positively interdependent preferences pertaining to the strict preference in (6) are:

$$\left.\begin{array}{cc} 1 \text{ to } k & (b, (be_k, ae_{n-k})) \sim (b, (be_k, ae_{n-k}) \\ k+1 & (b, (be_k, ae_{n-k})) \succ (a, (be_k, ae_{n-k})) \\ (k+2) \text{ to } n & (a, (be_{k+1}, ae_{n-k-1})) \sim (a, (be_{k+1}, ae_{n-k-1})). \end{array}\right\} \tag{7}$$

Note that the first and the third lines in (7) follows from reflexivity of $\succsim$ and the second line is the relation in (5). Since preferences are positively interdependent the orderings in (7) must imply (using (3)) $(a, (be_{k+1}, ae_{n-k-1})) \succ (b, (be_k, ae_{n-k}))$, in direct conflict with (6). This contradiction establishes that $x^{k+1} \succsim x^k$. In conclusion $x^k \succsim x^{k-1}$ for $k = 2, ..., n$ and $x^1 \succ x^0$. Now we can claim using transitivity that $(a, be_n) = x^n \succ x^0 = (a, ae_n)$ as was needed. ■

This lemma allows us to state a representation theorem based on assumptions made in this paper. It is shown that benevolent preferences satisfying **B1, B2, NE** and **SC** are representable.

Proposition 2: *If $\succsim$ is a benevolent preference order on $X = \mathbb{R}^{n+1}_{++}$ satisfying B1, B2, NE and SC, then $\succsim$ is representable.*

Proof: We will show that all conditions of Theorem 1 hold. Let us verify **DP**. Suppose $b > a$ we have to show that $(b, be_n) \succ (a, ae_n)$. By Lemma 1, we must have $(a, be_n) \succ (a, ae_n)$ and by **B1** we get $(b, be_n) \succ (a, be_n)$. So by transitivity it follows that $(b, be_n) \succ (a, ae_n)$ showing that **DP** is satisfied. Since **NE** and **SC** are assumed, we can invoke Theorem 1 to conclude that $\succsim$ is representable. ■

Remark 2:

(i) Comment on Lemma 1: One is tempted to try the proof of Lemma 1 in settling the open question that any benevolent preference order satisfying **B1, B2** must satisfy (4). However, to the best of our knowledge this method will fail. The proof of Lemma 1 goes through when every other person's income is identical because a particular symmetry is maintained when comparisons are made using (3) which is crucial to the proof. This symmetry is rendered invalid when the profiles x and y in (4) are non-constant.

(ii) Example of a Benevolent Preference order: The class of preference orders on X satisfying **B1, B2, NE** and **SC** is non-empty. Following Ok and Koçkesen (2000) we can define the following preference order and verify that all the conditions stated in Proposition 2 are met. Given $(a, x) \in X$, let $\mu(a, x)$ denote the average income of the society, so

$$\mu(a, x) = \frac{1}{n+1}(a + \sum_{i=1}^{n} x_i)$$

and define $\succsim_{rel}$ on X by $(a, x) \succsim_{+rel} (b, y)$ iff $a\mu(a, x) \geq b\mu(b, y)$. We can verify that $\succsim$ is positively interdependent. Suppose $x_i\mu(a, x) \geq y_i\mu(a, y)$ we need to show that $\mu(a, x) \geq \mu(a, y)$ holds to conclude $(a, x) \succsim_{+rel} (a, y)$. If $\mu(a, x) \geq \mu(a, y)$ fails, then $\mu(a, x) < \mu(a, y)$ which would imply $x_i \geq y_i$ for all i, since $x_i\mu(a, x) \geq y_i\mu(a, y)$ has to hold. Therefore, $a + \sum_i x_i \geq a + \sum_i y_i$ must be true which would yield $\mu(a, x) \geq \mu(a, y)$, contradicting the assumed converse. This shows that $a\mu(a, x) \geq a\mu(b, y)$ and hence, $(a, x) \succsim_{+rel} (a, y)$ holds proving that $\succsim_{+rel}$ is positively interdependent. **B1** and **B2** are easy to verify and we omit the details. To see **NE** and **SC** are satisfied, note that for each $(a, x) \in X$ the sets $A(x) = [\sqrt{a\mu(a, x)}, \infty)$ and $B(x) = (0, \sqrt{a\mu(a, x)}]$ are both non-empty and closed in $\mathbb{R}_{++}$. Observe that $\succsim_{+rel}$ satisfies (4) and thereby the Mitra-Ozbek result also guarantees the preference orders representability.

5 Conclusion

We have stated sufficient conditions (Diagonal Pareto, Nonemptiness and Scalar continuity) that guarantee representation of preference orders that are not necessarily monotone and demonstrated the usefulness of the result in two applications. In the first case we showed that for non-montone preferences our result can be used to obtain representation of preferences with a particular form of inequity aversion. The second example tackles the problem of representing benevolent preferences, where the non-monotonicity is an unresolved issue in the literature. However, we were able to show that degree of monotonicity we require (Diagonal Pareto) obtains which allows us to prove representation under conditions different from the ones pursued in the literature.

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