On Wold's Sufficiency Approach to Representation of Preferences

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Introduction

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- Chipman (1960): Countable order dense property "has little intuitive appeal".
- Wold (1943): Sufficient condition for Representability (Standard Textbook treatment of Representation MWG (1995), Varian (2014))

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  - each x is indifferent to at least one point on the diagonal (the 45° line).
  - under strong monotonicity, the intersection with the diagonal is unique.
  - Unique intersection with the diagonal serves as the utility function (A real-valued function u such that $u(x) \geq u(y)$ iff $x \succsim y$).

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  - Beardon and Mehta (1994), Mitra and Ozbek (2013), Banerjee (2014) recognize that monotonicity of some form is needed for the Wold technique.
  - We show that is not true.
  - Wold's method is about obtaining representation on a rich enough subset and then extending it to the whole space.

Theorem 1 (Framework)

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- Diagonal: $D = \{x \in X : \text{there is some } \lambda \in Y \text{ such that } x = \lambda e\}$ where $e = (1, 1, 1...)$
- Upper and Lower Contour sets restricted to the diagonal: $A(x) = \{\lambda \in Y : \lambda e \succsim x\}$ and $B(x) = \{\lambda \in Y : x \succsim \lambda e\}$

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- *Non-emptiness (NE)*: The sets $A(x)$ and $B(x)$ are non-empty subsets of Y for every $x \in X$.
- Theorem 1: Any preference order on a sequence space satisfying SC and NE is representable.

Steps of Proof of Theorem 1

- Under stated assumptions of SC and NE, the preference order restricted to the diagonal is representable (the set $Z = \{\lambda e \in D : \lambda$ rational$\}$ is order dense and countable). Denote this by $v : D \longrightarrow R$

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- The set $C(x) = A(x) \cap B(x) \neq \emptyset$.
- $u(x) = \{v(\lambda e) : \lambda \in C(x)\}$.

Violations of NE

- We can show that there is an example of a preference order on $X = [0, 1] \times [0, 1]$ which satisfies SC but NE *fails*, and the order is *not* representable.

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- Consider the sequence space $X = [0,1] \times [0,1]$ and let $Y = [0,1]$. Consider the following subsets of X: $V = \{x \in X : x_1 = 1\}$, $D = \{x \in X : \text{there is } \lambda \in Y \text{ such that } x = (\lambda, \lambda)\}$ and $R = X \setminus V \cup D$. Define a binary relation $\succsim$ on $X \times X$ as follows:

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  - (i) For any $x, y \in V \cup R$ we say $x \succsim y$ iff $x \geq_L y$ (where, $\geq_L$ is the standard lexicographic ordering), (ii) for any $x \in D$ and $y \in R$ we let $x \succ y$, (iii) for $[x \in V$ and $y \in D]$ or $[x \in D$ and $y \in V]$ or $[x, y \in D]$ we declare $x \succsim y$ iff $\min\{x_1, x_2\} \geq \min\{y_1, y_2\}$.

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- Theorem 1 can be strengthened by allowing NE to fail provided the violation is only on suitable countable partition (Theorem 2 in text)

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- A crucial property used in Theorem 1 and Wold (1943) is the non-empty intersection of $A(x) \cap B(x)$ for each x. Does postulating that $A(x) \cap B(x)$ is unique help us recover representation in the absence of SC?

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- Proposition 1 On every uncountable set there is a preference order that is not representable.
- By Proposition 1 there is binary relation P on $\mathbb{R}$, such that P a linear order (transitive and $x \neq y$ implies either $x \succ y$ or $y \succ x$) and the relation xRy iff xPy or $x = y$ is *not* representable. Now define the preference order $\succsim$ on $\mathbb{R}^2$ by

$$(x_1, x_2) \succsim (y_1, y_2) \text{ iff } (x_1 + x_2)R(y_1 + y_2).$$

Observe that

$$(x_1, x_2) \sim (\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2})$$

and for any $\lambda \neq [(x_1 + x_2)/2]$ we have $(x_1, x_2) \not\sim (\lambda, \lambda)$. Thus, $I(x)$ for any $x \in \mathbb{R}^2$ is a singleton. As R is not representable it must be the case that $\succsim$ is *not* representable either.

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Must representable preference orders necessarily have "rich" indifference classes?

- Intuitive criteria for representation and non-existence of representation in the lexicographic case might lead to the conjecture that lack of substitution possibilities has something to do with representation.
- This intuition is wrong.
- There is an example of a representable preference order which is strongly monotone, where each point is only indifferent to itself yet the order is representable.

Example

- Consider the sequence space $X = Y^2$ with $Y = [0, 1]$. For any $p \in Y$ we can express p in its binary expansion form as follows:

$$p = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots + \frac{a_n}{2^n} + \cdots \quad (1)$$

where $a_i \in \{0, 1\}$ for each $i \in \mathbb{N}$. This representation is unique unless p is of the form $q/2^n$ where $0 < q < 2^n$ is an integer, in which case there is precisely two representations.

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- Now define $f : Y \to Y$ by ($\{a_i\}$ is the unique representation as above) $f(p) = \frac{a_1}{4} + \frac{a_2}{4^2} + \frac{a_3}{4^3} + \cdots + \frac{a_n}{4^n} + \cdots$ and $h : Y \to Y$ by $h(q) = f(q)/2$. Finally define $u : X \to \mathbb{R}$ by
  $u(x_1, x_2) = f(x_1) + h(x_2)$.

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- Using u define the binary relation on $X \times X$ as:

$$\text{for all } x, y \in X \text{ we will say } x \succsim y \text{ iff } u(x_1, x_2) \geq u(y_1, y_2).$$

Example Continued

- Given $p, p' \in [0,1]$ let $m(p,p')$ denote the $\min\{i \in \mathbb{N} : a_i \neq a_i'\}$, where $\{a_i\}$and $\{a_i'\}$ are the standard binary expansions of p and p'.

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- **Observation** *Let* $1 \geq p' > p \geq 0$ *and* $\{a_i\}$ *and* $\{a_i'\}$ *be the standard binary expansions of* p *and* p' respectively. *Denote* $m(p,p')$ *by* r. *Then (a)* $a_r' = 1, a_r = 0$ *(b) there is some* $n \geq (r+1)$ *such that* $(a_n' - a_n) \in \{0,1\}$ *(c)* $\frac{(4/3)}{4^r} \geq [f(p') - f(p)] \geq \frac{(2/3)}{4^r}$.

Example Continued

- Given $p, p' \in [0,1]$ let $m(p,p')$ denote the $\min\{i \in \mathbb{N} : a_i \neq a'_i\}$, where $\{a_i\}$and $\{a'_i\}$ are the standard binary expansions of p and p'.
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- Observation implies that if $x \neq x'$ we must have $u(x) \neq u(x')$.

- Thank you!